ON THE COHOMOLOGY RINGS OF HAMILTONIAN T-SPACES

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ABSTRACT. Let M be a symplectic manifold equipped with a Hamiltonian action of a torus T. Let F denote the fixed point set of the T-action and let $i : F \hookrightarrow M$ denote the inclusion. By a theorem of F. Kirwan [K] the induced map $i^* : H_T^*(M) \longrightarrow H_T^*(F)$ in equivariant cohomology is an injection. We give a simple proof of a formula of Goresky-Kottwitz-MacPherson [GKM] for the image of the map i^*.

1. INTRODUCTION

The classification of manifolds equipped with group actions presents difficulties beyond those inherent in the study of manifolds per se. Even the basic questions—What is the equivariant cohomology ring of the manifold? What can be said about the fixed manifolds of the group action? What is the cohomology ring of the quotient?—turn out to be delicate and involved.

Much more is known in the case of a symplectic manifold (M^{2m}, ω) equipped with a Hamiltonian action of a torus T. Let $H_T^*(M)$ denote the rational equivariant cohomology ring of M.

The following theorem of F. Kirwan relates the equivariant cohomology of M with the equivariant cohomology of its fixed point set:

Theorem 1.1. *(Kirwan)*[K] *Let a torus T act on a compact symplectic manifold (M, ω) in a Hamiltonian fashion. Denote the fixed point set of the action by F. The natural inclusion $i : F \longrightarrow M$ of the fixed point set in the manifold induces an injection $i^* : H_T^*(M) \hookrightarrow H_T^*(F)$.*

A related result, also due to Kirwan, relates the equivariant cohomology of M and the cohomology of the symplectic quotient of M.

S. Tolman was partially supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship, by an NSF grant, and by an Alfred P. Sloan Foundation Fellowship. J. Weitsman was supported in part by NSF grant DMS 94/03567, by NSF Young Investigator grant DMS 94/57821, and by an Alfred P. Sloan Foundation Fellowship.

February 8, 2008.

Theorem 1.2. *(Kirwan)*[K] *Let a torus T act on a compact symplectic manifold (M, ω) with moment map $\mu : M \longrightarrow \mathfrak{t}^* = \mathrm{Lie(T)}^*$. Suppose that $\xi \in \mathfrak{t}^*$ is a regular value of μ and let $M_\xi := \mu^{-1}(\xi)/T$ be the symplectic quotient of M. The inclusion map $K : \mu^{-1}(\xi) \longrightarrow M$ induces a surjection $K^* : H_T^*(M) \longrightarrow H_T^*(\mu^{-1}(\xi)) = H^*(M_\xi)$.*

These two results give rise to two natural questions. What is the image of i^*? And what is the kernel of K^*? The purpose of this paper is to address the first question. In a companion paper [TW], we answer the second.

The description we give of the image of the injective map i^* is due to Goresky, Kottwitz, and MacPherson [GKM], using the work of Chang and Skjelbred ([CS]; see also [AP, BV, H]).

Definition 1.3. Let $N \subset M$ denote the subset of M consisting of points whose T-orbits are one-dimensional; i.e.,

$$N := \{p \in M \mid T \cdot p \sim S^1\}$$

By the local normal form theorem, each connected component N_α of N is an open symplectic manifold whose closure $\overline{N_\alpha}$ is a compact, symplectic submanifold of M. The restriction of μ to $\overline{N_\alpha}$ is a moment map for the restricted torus action. Furthermore the closure of N is given by $\overline{N} = N \cup F$; this is referred to as the **one-skeleton** of M.

Example 1.4. Consider the case where M^{2m} is a $2m$ dimensional toric variety, equipped with the appropriate Hamiltonian action of a torus T of rank m. The image of the moment map is the moment polytope $\Delta = \mu(M)$. Let $v(\Delta)$ denote the union of of the vertices of Δ, and $e(\Delta)$ the union of the interior of the edges. Then $F = \mu^{-1}(v(\Delta))$, while $N = \mu^{-1}(e(\Delta))$.

The main result of this paper is the following

Theorem 1. *Let (M, ω) be a compact symplectic manifold equipped with a Hamiltonian action of a torus T. Let F be the fixed point set and let $\overline{N}$ be the one-skeleton. Let $i : F \longrightarrow M$ and $j : F \longrightarrow \overline{N}$ be the natural inclusions, and $H_T^*(M) \xrightarrow{i^*} H_T^*(F)$ and $H_T^*(\overline{N}) \xrightarrow{j^*} H_T^*(F)$ be the pull-back maps in equivariant cohomology. Then the images of i^* and j^* are the same.*

This theorem is proved in considerable generality in [GKM]. The purpose of this paper is to give a simple proof of this result in the symplectic setting, which will enable us to obtain a description of the cohomology ring of M in a form that will make the structure of the

map K^* to the cohomology ring of the symplectic quotient transparent. Our methods should yield similar statements in integral cohomology. Additionally, our methods give an algorithm for turning this description of the cohomology ring into an explicit set of generators and relations.

Before outlining the proof of Theorem 1, let us consider a few special examples.

Example 1.5. If the torus T is one-dimensional, $\overline{N} = M$, so the theorem is obviously true but trivial.

Example 1.6. Suppose that the closure $\overline{N_i}$ of each of the components of N is a copy of the two-sphere P^1. For each such component there exists a corank-one subgroup $K_i \subset T$ which acts trivially on $\overline{N_i}$. The quotient T/K_i is isomorphic to S^1, and the corresponding action on P^1 must be the usual action, so that $H_T^*(\overline{N_i})$ is given as follows. Let $\gamma_i : K_i \longrightarrow T$ denote the inclusion, and let $\gamma_i^* : H_T^*(\mathrm{pt}) \longrightarrow H_{K_i}^*(\mathrm{pt})$ denote the induced map in equivariant cohomology. For each i, the set $\overline{N_i} \cap F$ consists of two points n_i, s_i; and the image of $H_T^*(\overline{N_i})$ in $H_T^*(\overline{N_i} \cap F) = H_T^*(n_i) \oplus H_T^*(s_i)$ consists of those elements $(a, b) \in H_T^*(n_i) \oplus H_T^*(s_i)$ such that $\gamma_i^* a = \gamma_i^* b$. Let the fixed points of the T-action be given by F_i, $i = 1, \dots, N$. Then the image of $H_T^*(M)$ in $H_T^*(F) = \bigoplus H_T^*(F_i)$ consists of

$$(a_1, \dots, a_N) \in H_T^*(F_1) \oplus \dots H_T^*(F_N) \tag{1.7}$$

such that, for each $\overline{N_i}$,

$$\gamma_i^* a_{n_i} = \gamma_i^* a_{s_i}$$

(See [GKM], [GS]). This gives a completely combinatorial algorithm for computing $H_T^*(M)$.

The main tool needed to prove Theorem 1, as well as the injectivity and surjectivity theorems 1.1, 1.2, is a repeated use of equivariant Morse theory. The key fact in all these cases is that components of the moment map μ give equivariantly self-perfecting Morse functions whose critical set is precisely F. As the same is true for each of the $\overline{N_i}$'s, similar statements can be made for the one-skeleton $\overline{N}$.

These self-perfecting Morse functions give us a very useful way of constructing the equivariant cohomology ring of M from the cohomology rings of the fixed manifolds F_i: roughly speaking, the contribution of each fixed manifold to the cohomology ring of M consists of those classes in $H_T^*(M)$ which vanish on all fixed points "below" F_i, and whose value on F_i is a multiple of the downward Euler class of the

Morse flow. As a similar statement can be made about the cohomology ring of $\overline{N}$, we may compare the images of $H_T^*(M)$ and $H_T^*(\overline{N})$ to prove our result.

2. Morse Theory and the Moment Map

In this section we state several results which will be the key steps in the proof of Theorem 1. Among them is Kirwan's injectivity theorem (of which we supply a proof). All of these results follow directly from the equivariant Morse complex associated to the choice of a Hamiltonian as a Morse function. We note that several of the results of this section have analogs in integral cohomology; however we are only concerned with rational cohomology in this paper.

Let us recall our set-up. Let (M, ω) be a compact symplectic manifold with a moment map μ for the action of a torus T^n. Let $F \subset M$ denote the fixed point set of the torus action. Given a generic element $\xi \in \mathfrak{t}$, the function $f = \langle \mu, \xi \rangle : M \longrightarrow \mathbf{R}$ is a Morse function on M whose critical set coincides with the fixed point set F.

Let us consider the fundamental exact sequence corresponding to the Morse function f. Let us denote by C the critical set of f, and choose $c \in C$. We may assume that an interval $[c-\epsilon, c+\epsilon]$ contains no critical values of f other than c. Let $M_c^+ = f^{-1}(-\infty, c+\epsilon)$, $M_c^- = f^{-1}(-\infty, c-\epsilon)$. Then we have the following lemma, which is the main technical fact behind our results:

Proposition 2.1. *Let a torus T act on a compact manifold M with moment map $\mu : M \longrightarrow \mathfrak{t}^*$. Given $\xi \in \mathfrak{t}$, choose a critical value c of the projection $f := \mu^\xi$. Let F be the set of fixed points, and let F_c be the component of F with value c.*

The long exact sequence in equivariant cohomology for the pair (M_c^+, M_c^-) splits into short exact sequences:

$$0 \longrightarrow H_T^*(M_c^+, M_c^-) \longrightarrow H_T^*(M_c^+) \xrightarrow{k^*} H_T^*(M_c^-) \longrightarrow 0. \tag{2.2}$$

Moreover, the restriction from $H_T^*(M_c^+)$ to $H_T^*(F_c)$ induces an isomorphism from the kernel of k^* to those classes in $H_T^*(F_c)$ which are multiples of e_c, the equivariant Euler class of the negative normal bundle of F_c.

Proof: By our assumptions, f is a Morse function, and there is a unique critical value of f contained in the interval $[c-\epsilon, c+\epsilon]$. Denote the corresponding critical manifold by F_c, and the negative disc and sphere bundles of F_c by D_c, S_c respectively. The pair (M_c^+, M_c^-) can be retracted onto the pair (D_c, S_c), so there is an isomorphism

$$H_T^*(M_c^+, M_c^-) \cong H_T^*(D_c, S_c) \tag{2.3}$$

By the Thom isomorphism theorem, we have

$$H_T^*(D_c, S_c) \cong H_T^{*-\lambda_c}(D_c) \ = H_T^{*-\lambda_c}(F_c) \tag{2.4}$$

where λ_c is the Morse index of the critical manifold F_c; so we obtain a commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & H_T^*(M_c^+, M_c^-) \arrow[r, "\gamma_c"] \arrow[d] & H_T^*(M_c^+) \arrow[r, "\beta_c"] \arrow[d] & H_T^*(M_c^-) \arrow[r] & {} \\
& H_T^*(D_c, S_c) \arrow[r, "\delta_c"] \arrow[d] & H_T^*(D_c) & & \\
& H_T^{*-\lambda_c}(D_c) \arrow[ur, "\cup e_c"'] & & &
\end{tikzcd} \tag{2.5}$$

where $e_c = e(D_c)$ is the equivariant Euler class of the bundle $D_c \longrightarrow F_c$. The cup product map $\cup e_c$ is *injective*; the same therefore is true of the maps δ_c and γ_c, proving the lemma.

The following corollary is then immediate:

Corollary 1. *(see* [AB1, AB2, K]*) The function f is an equivariantly perfect Morse function on M.*

Another application of this proposition is the proof of the following theorem of Kirwan.

Theorem 2. *Let a torus T act on a symplectic manifold (M, ω) with proper bounded below moment map $\mu : M \longrightarrow \mathfrak{t}^*$. Let $i : F \longrightarrow M$ denote the natural inclusion of the set F of fixed points. The pullback map $i^* : H_T^*(M) \longrightarrow H_T^*(F)$ is injective.*

Proof. Order the critical values of f as $c_1 < c_2 < \cdots < c_n$. The theorem obviously holds for $f^{-1}(-\infty, c_1) = \emptyset$. Assume the proposition holds for the manifold $M^- := f^{-1}(-\infty, c_i)$. We will show that it will hold for the manifold $M^+ := f^{-1}(-\infty, c_{i+1})$; the result follows then by induction.

By Lemma 2.1, we have a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H_T^*(M^+, M^-) \arrow[r] \arrow[d, "\simeq"] & H_T^*(M^+) \arrow[r] \arrow[d, "i_+^*"] & H_T^*(M^-) \arrow[r] \arrow[d, "i_-^*"] & 0 \\
0 \arrow[r] & H_T^*(F_i) \arrow[r] & H_T^*(F \cap M^+) \arrow[r] & H_T^*(F \cap M^-) \arrow[r] & 0,
\end{tikzcd} \tag{2.6}$$

where F_i denotes the critical set with value c_i. By induction, the inclusion i_- of $F \cap M^-$ into M^- induces an injection in cohomology. By Proposition 2.1 the image of $H_T^*(M^+, M^-)$ in $H_T^*(M^+)$ is embedded injectively in $H_T^*(F_i)$. The theorem then follows by diagram chasing. □

3. Proof of the Main Theorem

We are now ready to prove our main theorem: the restriction map to the fixed point set induces an isomorphism from the equivariant cohomology of the original Hamiltonian manifold to the image of the equivariant cohomology of the one-skeleton, under its restriction map to the fixed point set. The key idea is to use the tools developed in the last section to compare the graded rings associated to these images using the filtration given by the Morse function obtained by choosing a projection of the moment map. The result will then follow from the naturality of these objects, induction on the critical points, and the injectivity of the restriction map.

Recall that the one-skeleton is given by

$$\overline{N} := \{p \in M \mid T \cdot p \text{ is one-dimensional or zero-dimensional}\}.$$

Clearly, the image of $i^* : H_T^*(M) \longrightarrow H_T^*(F)$ is a subset of the image of $j^* : H_T^*(\overline{N}) \longrightarrow H_T^*(F)$. Therefore, i^* induces a map, which we will also call i^*, from $H_T^*(M)$ to $\operatorname{im} j^* \subset H_T^*(F)$. By Theorem 1.1, this map is injective. Therefore, to prove the theorem, it suffices to show that this map is surjective.

On the level of the graded rings associated to the Morse filtration, surjectivity will follow from comparing the proposition below with Proposition 2.1.

Proposition 3.1. *Let a torus T act on a compact symplectic manifold M with moment map $\mu : M \longrightarrow \mathfrak{t}^*$. Given $\xi \in \mathfrak{t}$, choose a critical point c of the projection $f := \mu^\xi$. Let F denote the fixed point set and let F_c denote the component of F with value c. Define $F^- := F \cap f^{-1}(-\infty, c - \epsilon)$ and $\overline{N^+} := \overline{N} \cap f^{-1}(-\infty, c + \epsilon)$.*

Let η be a cohomology class in $H_T^*(\overline{N^+})$ which vanishes when restricted to $H_T^*(F^-)$. Its restriction to $H_T^*(F_c)$ is a multiple of $e_c =$

$e(D_c)$, the equivariant Euler class of the downward normal bundle D_c of F_c (in M).

Proof. Consider any component N_α of the set N of one-dimensional orbits such that the closure $\overline{N_\alpha}$ contains F_c. The closure $\overline{N_\alpha}$ is a smooth T-invariant symplectic manifold with moment map μ. The class η induces a cohomology class on $\overline{N_\alpha^+} := \overline{N_\alpha} \cap f^{-1}(-\infty, c+\epsilon)$ which vanishes when restricted to $\overline{N_\alpha} \cap F^-$, and hence (by injectivity which we need to state in this version), when restricted to $\overline{N_\alpha^-} := \overline{N_\alpha} \cap f^{-1}(-\infty, c-\epsilon)$. Thus, by Proposition 2.1, any element of the kernel of the natural map $H_T^*(N_\alpha^+) \longrightarrow H_T^*(N_\alpha^-)$ is, when restricted to $H_T^*(F_c)$, a multiple of the equivariant Euler class e_α (here $e_\alpha = e(D_c \cap \overline{N_\alpha})$ is the equivariant Euler class e_c of the downward normal bundle $D_c \cap \overline{N_\alpha}$ of F_c in $\overline{N_\alpha}$).

So the restriction of η to F_c is a multiple of the equivariant Euler class of the downward normal bundle of F_c in N_α. Since this holds for each component N_α, and each of these components must have a different stabilizer, we may apply Lemma 3.2 below. Therefore, the class η must be multiple of the product of the equivariant Euler classes the negative normal bundles to F_c in all the components of N whose closure contains F_c. But this is precisely the equivariant Euler class of the negative normal bundle to F_c in M. □

Lemma 3.2. *Let a torus T act on a complex vector bundle E over a manifold F, so that the fixed set is precisely F. Decompose E into the direct sum of bundles E_α, where each E_α is acted on with a different weight $\alpha \in \mathfrak{t}^*$. Let e_α be the Euler class of the sub-bundle E_α.*

Then if $y \in H_T^*(F)$ is a multiple of e_α for each α, then y is a multiple of the product of the e_α.

Proof. Assume first that F is a single point.

Let $\alpha \in \mathfrak{t}^*$ be the weight with which T acts on the sub-bundle E_α. Since α is a linear function on $\mathfrak{t}$, it lies naturally in $H^*(BT) = H_T^*(F) = \mathrm{Sym}(\mathfrak{t}^*)$, the algebra of symmetric polynomials on $\mathfrak{t}$. The equivariant euler class of E_α is given by $e_\alpha = \alpha^{n_\alpha}$, where n_α is the complex dimension of E_α. The α are distinct by assumption, and non-zero since no point not in the zero section is fixed by T. Therefore, the e_α are pairwise relatively prime. (Recall that every polynomial ring over $\mathbf{Q}$ is a unique factorization domain.)

More generally, since F is fixed by T, $H_T^*(F) = H^*(F) \otimes H^*(BT)$. Thus, $H_T^*(F)$ is bigraded. In particular, given any integer i, any cohomology class $a \in H_T^*(F)$ has a well-defined component $a_i \in H^i(F) \otimes$

$H^*(BT)$, and the sum of all such components is a itself; we will call a_i **the component of a with F-degree i.**

Note that the component of e_α with F-degree 0 is precisely α^{n_α}. By the previous discussion, these are non-zero and pairwise relatively prime. Therefore it is enough to prove that if e and f are two cohomology classes whose components e_0 and f_0 with F-degree zero are relatively prime, and if e and f both divide α, then so does $e \cdot f$. We will prove this by induction.

We claim that if $e(f \cdot w + x) = f(e \cdot w + y)$, where the components of x and y with F-degree i vanish for all $i < k$, then there exist x', y' such that $e(f \cdot w + x') = f(e \cdot w + y')$, and such that the components of x' and y' with F-degree i vanish for all $i < k+1$. To see this, compare the component of F-degree k on the two sides of the equation $e(f \cdot w + x) = f(e \cdot w + y)$. Cancelling out terms which appear on both sides, we get $e_0 x_k = f_0 y_k$. Since e_0 and f_0 are relatively prime polynomials, this shows that there exists z_k such that $x_k = f_0 z_k$ and $y_k = e_0 z_k$. $\square$

We now proceed to prove that the map $i^* : H_T^*(M) \longrightarrow \operatorname{im} j^* \subset H_T^*(F)$ is surjective. We proceed, as usual, by induction:

Consider any critical point c of the projection $f := \mu^\xi$. Define $M^+ := f^{-1}(-\infty, c+\epsilon)$, and $M^- := f^{-1}(-\infty, c-\epsilon)$ for any sufficiently small ϵ. Let $\overline{N^+} := \overline{N} \cap M^+$, $\overline{N^-} := \overline{N} \cap M^-$, $F^+ := F \cap M^+$, and $F^- := F \cap M^-$. Let $i^+ : F^+ \longrightarrow M^+$, $i^- : F^- \longrightarrow M^-$, $j^+ : F^+ \longrightarrow \overline{N^+}$ and $j^- : F^- \longrightarrow \overline{N^-}$ denote the corresponding inclusion maps. It is enough to assume that the induced map $i^{-*} : H_T^*(M^-) \longrightarrow \operatorname{im} j^{-*} \subset H_T^*(F^-)$ is surjective, and to prove that the induced map $i^{+*} : H_T^*(M^+) \longrightarrow \operatorname{im} j^{+*} \subset H_T^*(F^+)$ is also surjective.

Since the images of $H_T^*(M^-)$ and $H_T^*(N^-)$ inside $H_T^*(F^-)$ are the same, it follows that the natural restriction map r from $\operatorname{im} j^{+*} \subset H_T^*(F^+)$ to $\operatorname{im} j^{-*} \subset H_T^*(F^-)$ is surjective. Thus, taking the exact sequence (2.2) of Proposition 2.1, we have a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H_T^*(M^+, M^-) \arrow[r] \arrow[d] & H_T^*(M^+) \arrow[r] \arrow[d, "i_+^*"] & H_T^*(M^-) \arrow[r] \arrow[d, "i_-^*"] & 0 \\
0 \arrow[r] & \ker r \arrow[r] & \operatorname{im} j^{+*} \subset H_T^*(F^+) \arrow[r, "r"] & \operatorname{im} j^{-*} \subset H_T^*(F^-) \arrow[r] & 0,
\end{tikzcd} \tag{3.3}$$

By our inductive assumption, i^{-*} is surjective.

By Proposition 3.1, every element in $\ker r$ is a multiple of e_c, the equivariant Euler class of the negative normal bundle of F_c, the component of the fixed point set with value c. On the other hand, by Proposition 2.1, every multiple of e_c is in the image of the restriction $H_T^*(M^+, M^-)$ to $H_T^*(F_c)$ Thus, the arrow from $H_T^*(M^+, M^-)$ to $\ker r$ is surjective too.

The result follows by a diagram chase.

4. Some comments about integral cohomology

We close with some comments about integral cohomology. Unfortunately, the integer version of Theorem 1 is not true in general. In fact even the injectivity theorem 1.1 will not hold in integral cohomology without some assumptions. However, both injectivity and a version of Theorem 1 can be proved over the integers where certain restrictions are placed on the allowable stabilizer subgroups.

Perhaps the easiest example where Theorem 1 is not true for integer cohomology is that of $S^1 \times S^1$ acting on $S^2 \times S^2$, with speed two on each sphere.[1] Using the moment map for the diagonal action as our Morse function, we see that every cohomology class which vanishes outside a neighborhood of the north pole $\times$ the north pole must be a multiple of $4x^2$ when restricted to that point. In contrast, there exists a cohomology class on the one-skeleton which vanishes outside this neighborhood but is only a multiple of $2x^2$. Essentially, the problem is that the weights are not relatively prime. It is easy to place a condition on the stabilizer groups at each fixed point in a way that negates this possibility. This is essentially all that can go wrong, and a version of Theorem 1 can be expected to hold if such an assumption of relative primality is made.

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[1] If the reader is disturbed by the fact that this action is not effective, she or he may tack on another couple of S^2's spinning at speed one.

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